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THE INTERACTION BETWEEN A PLANE INCLINED RING-SHAPED PUNCH AND AN ELASTIC HALF-SPACE†

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Solutions of the problem of the action of an inclined ring punch on an elastic half-space are constructed using asymptotic methods [1–3], with the usual assumptions that there are no friction forces in the contact region between the punch and the half-plane and that the surface of the half-plane outside the contact area is not loaded. The solutions are obtained in the form of simple formulae for large and small values of the dimensional parameter λ, which represents the relative thickness of the ring. These solutions overlap one another with a high degree of accuracy in a certain intermediate range of variation of λ.

This problem has been investigated previously by many researchers in a similar formulation. In particular, we draw attention to the papers [4–6]. Here we obtain a comprehensive solution of the problem, enabling a complete qualitative and quantitative analysis of the problem to be carried out.

1. According to the classical scheme (see, for example, [7]) the problem of a ring punch (see Fig. 1) can be reduced to determining the contact pressures $q(r, \phi)$ from the integral equation

$$\int_a^b \int_0^{2\pi} \frac{q(\rho,\psi)\rho d\rho d\psi}{R} = 2\pi\theta\delta(r,\phi)$$

$$(R = [r^2+\rho^2-2r\rho\cos(\phi-\psi)]^{1/2},\quad \theta = G/(1-\nu),\quad a \leqslant r \leqslant b,\quad \phi\in[0,2\pi)) \tag{1.1}$$

where a and b are the inner and outer radii of the ring region of the contact, $\delta(r, \phi)$ is the residue of points of the surface of the elastic half-space in the contact region, and G and ν are elasticity constants.

Further, using the integral [8, 6.511(1)] and relation [8, 8.531(1)]

$$J_0(uR) = J_0(ur)J_0(u\rho) + 2\sum_{m=1}^{\infty} J_m(ur)J_m(u\rho)\cos m(\phi-\psi) \tag{1.2}$$

where $J_m(x)$ are Bessel functions, we can reduce Eq. (1.1) to the form

$$\int_a^b \int_0^{2\pi} q(\rho,\psi)[k_0(r,\rho)+2\sum_{m=1}^{\infty} k_m(r,\rho)\cos m(\phi-\psi)]\rho d\rho d\psi = 2\pi\theta\delta(r,\phi) \tag{1.3}$$

where the kernels $k_m(r, \rho)$ have the form

$$k_m(r,\rho) = \int_0^{\infty} J_m(ur)J_m(u\rho)du \tag{1.4}$$

Using the integral [8, 6.512(1)] and one of the identical transformations of a hypergeometric series [8, 9.134(3)], we will have

$$k_m(r,\rho) = \frac{(2m-1)!!e^{2m}}{2^{2m}(2m)!!(r+\rho)} F\left(m+\frac{1}{2},\ m+\frac{1}{2},\ 2m+1, e^2\right) \quad \left(e = \frac{2\sqrt{r\rho}}{r+\rho}\right) \tag{1.5}$$

where $(F(\alpha, \beta, \gamma, x)$ is the hypergeometric series. Finally, using the integral representation of a hypergeometric series [8, 9.111], we can rewrite (1.5) in the form

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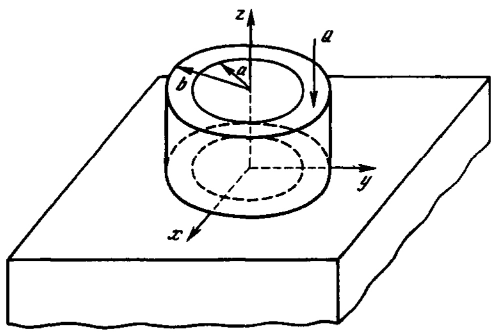

Fig. 1.

$$k_m(r,\rho)=\frac{e^{2m}}{\pi(r+\rho)}\int_0^1 t^{m-\frac{1}{2}}(1-t)^{m-\frac{1}{2}}(1-e^2t)^{-m-\frac{1}{2}}dt \tag{1.6}$$

For any m the integral in (1.6) can be expressed by a linear combination of complete elliptic integrals $K(e)$ and $E(e)$. For example, for $m = 0$ and $m = 1$ we have, respectively

$$k_0(r,\rho)=\frac{2}{\pi(r+\rho)}K(e),\quad k_1(r,\rho)=\frac{2}{\pi(r+\rho)e^2}[(2-e^2)K(e)-2E(e)] \tag{1.7}$$

It is important to note that the kernels $k_m(r, \rho)$ can again be represented by the integral

$$k_m(r,\rho)=\frac{1}{\pi\sqrt{r\rho}}\int_0^\infty \frac{L_m(s)}{s}\cos\left(s\ln\frac{r}{\rho}\right)ds$$
$$L_m(s)=\frac{s\Gamma(\frac{1}{4}+is/2+m/2)\Gamma(\frac{1}{4}-is/2+m/2)}{2\Gamma(\frac{3}{4}+is/2+m/2)\Gamma(\frac{3}{4}-is/2+m/2)} \tag{1.8}$$

where $\Gamma(z)$ is the gamma function. Here we have used the relation

$$\int_0^\infty J_m(t)J_m\left(t\frac{r}{\rho}\right)dt=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s)\left(\frac{r}{\rho}\right)^{-s}ds$$
$$g(s)=\frac{\Gamma(s/2+m/2)\Gamma(\frac{1}{2}-s/2+m/2)}{2\Gamma(1-s/2+m/2)\Gamma(\frac{1}{2}+s/2+m/2)} \tag{1.9}$$

obtained using the formulae for the direct and inverse Mellin transformation of a Bessel function [9].

We expand the functions $q(\rho, \psi)$ and $\delta(r, \phi)$ in (1.3) in Fourier series

$$q(\rho,\psi)=q_0(\rho)+2\sum_{n=1}^\infty [q_n(\rho)\cos n\psi+\tilde{q}_n(\rho)\sin n\psi]$$
$$\delta(r,\phi)=\delta_0(r)+2\sum_{n=1}^\infty [\delta_n(r)\cos n\phi+\tilde{\delta}_n(r)\sin n\phi] \tag{1.10}$$

Then, evaluating the integral over ψ in (1.6) and equating the coefficients of similar $\cos n\varphi$ and $\sin n\varphi$ terms on the right and left we obtain

$$\pi\int_a^b q_n(\rho)k_n(r,\rho)\rho d\rho=\theta\delta_n(r),\cdot\int_a^b \tilde{q}_1(\rho)k_1(r,\rho)\rho d\rho=\theta\tilde{\delta}_1(r)\ (n=0,1) \tag{1.11}$$

and so on. For a plane inclined ring punch $\delta_0(r) = \delta$, $\delta_1(r) = \alpha r$, $\tilde{\delta}_1(r) = \beta r$, all the remaining $\delta_n(r)$ and

$\tilde{\delta}_n(r)$ are equal to zero, and all the $q_n(r)$ and $\tilde{q}_n(r)$ will respectively be equal to zero for $n \geqslant 2$. Here δ is the translational displacement of the punch along the z axis, and α and β are its angles of rotation with respect to the y and x axes.

Thus, the problem of a plane inclined ring punch has been reduced to solving integral equations (1.11). To complete the formulation of the problem we need to add the conditions of equilibrium of the punch

$$Q = 2\pi\int_a^b q_0(\rho)\rho d\rho$$

$$M_x = Qe_y = \pi\int_a^b \tilde{q}_1(\rho)\rho^2 d\rho, \quad M_y = Qe_x = \pi\int_a^b q_1(\rho)\rho^2 d\rho \tag{1.12}$$

where e_x and e_y are the projections of the point of application of the impressing force Q onto the x and y axes. Conditions (1.12) serve to determine the relationships between Q and δ, e_x and α, e_y and β.

2. We will first consider the case of a relatively narrow ring. We make the following change of variables in the integral equations (1.11)

$$r = a\exp[(1+x)/\lambda], \quad \rho = a\exp[(1+\xi)/\lambda], \quad \lambda = 2[\ln(b/a)]^{-1} \tag{2.1}$$

and introduce the following notation

$$p_0(\xi) = \frac{\rho^{3/2}q_0(\rho)}{\theta\delta a^{1/2}}, \quad p_1(\xi) = \frac{\rho^{3/2}q_1(\rho)}{\theta\alpha a^{3/2}} = \frac{\rho^{3/2}\tilde{q}_1(\rho)}{\theta\beta a^{3/2}}$$

$$M_0(t) = \operatorname{sch}\frac{t}{2}K\left[\operatorname{sch}\frac{t}{2}\right]\left(t = \frac{x-\xi}{\lambda}\right) \tag{2.2}$$

$$M_1(t) = \left\{\left[2-\operatorname{sch}^2\frac{t}{2}\right]K\left[\operatorname{sch}\frac{t}{2}\right] - 2E\left[\operatorname{sch}\frac{t}{2}\right]\right\}\left[\operatorname{sch}\frac{t}{2}\right]^{-1}$$

where sch(x) is the hyperbolic secant. Equations (1.11) can then be rewritten in the form

$$\int_{-1}^{1} p_n(\xi)M_n\left(\frac{x-\xi}{\lambda}\right)d\xi = \pi\lambda\exp\left[\left(\frac{1}{2}+n\right)\frac{(1+x)}{\lambda}\right] \quad (n=0,1) \tag{2.3}$$

It is important to note that the kernels $M_0(t)$ and $M_1(t)$ of Eqs (2.3) can be represented using expressions (2.2) and expansions of the complete elliptic integrals for values of the argument close to unity ([8, 8.113(3)] and [8.8.114(3)]), in the same form

$$\sum_{i=0}^{\infty} a_i t^{2i} + \ln|t|\sum_{i=0}^{\infty} b_i t^{2i} \tag{2.4}$$

where the series converge absolutely when $|t| < \pi$. Here for $M_0(t)$ we have $a_0 = 2.079$, $a_1 = -0.1091$, $a_2 = 0.005352$, $b_0 = -1$, $b_1 = 0.06250$ and $b_2 = -0.003581$, and for $M_1(t)$ we have $a_0 = 0.07944$, $a_1 = 0.2857$, $a_2 = 0.004494$, $b_0 = -1$, $b_1 = -0.1875$ and $b_2 = 0.0009766$. Hence, further consideration can lead to the single equation

$$\int_{-1}^{1} \varphi(\xi)\left[\sum_{i=0}^{\infty} a_i\left(\frac{x-\xi}{\lambda}\right)^{2i} + \ln\left|\frac{x-\xi}{\lambda}\right|\sum_{i=0}^{\infty} b_i\left(\frac{x-\xi}{\lambda}\right)^{2i}\right]d\xi = \lambda\exp\left[\frac{\mu(1+x)}{\lambda}\right] \tag{2.5}$$

where $\varphi(\xi) = p_0(\xi)$, $\mu = 1/2$ in the case of the first equation of (2.3) and $\varphi(\xi) = p_1(\xi)$, $\mu = 3/2$ in the case of the second equation of (2.3).

For sufficiently large values of the dimensionless parameter λ ($\lambda > 2/\pi$), i.e. for a relatively narrow ring, the asymptotic solution of integral equation (2.5) can be constructed by the standard method described, for example, in [3]. We will give the final formulae

$$\varphi(x)=\varphi_0(\lambda,x)+\lambda^{-2}\varphi_1(\lambda,x)+\lambda^{-4}\varphi_2(\lambda,x)+O(\lambda^{-6})$$

$$\varphi_0(\lambda,x)=\frac{1}{\sqrt{1-x^2}}\left\{\frac{P}{\pi}-[c_0xQ_0+2c_1Q_1+3c_2xQ_1+4c_3Q_2+5c_4xQ_2+6c_5Q_3]\right\}$$

$$\varphi_1(\lambda,x)=\frac{1}{\sqrt{1-x^2}}\left\{\frac{2P}{\pi}A\left(\frac{3}{2}\right)Q_1-c_0[A(1)xQ_0+b_1xQ_1]+c_1b_1\left(Q_1-\frac{2}{3}Q_2\right)-\right.$$
$$\left.-c_2\left[\frac{3}{4}A\left(\frac{5}{4}\right)xQ_0+\frac{1}{2}b_1xQ_2\right]+c_3b_1\left(\frac{3}{4}Q_1-\frac{2}{5}Q_3\right)\right\}$$

$$\varphi_2(\lambda,x)=\frac{1}{\sqrt{1-x^2}}\left\{\frac{4P}{\pi}\left[B\left(\frac{25}{12}\right)Q_2+\frac{3}{2}B\left(\frac{13}{12}\right)Q_1+\frac{1}{2}b_1A\left(\frac{3}{2}\right)\left(-\frac{1}{2}Q_1+\frac{1}{3}Q_2\right)\right]-\right.$$
$$-c_0\left\langle\frac{3}{2}B\left(\frac{5}{6}\right)xQ_0+6B\left(\frac{19}{20}\right)xQ_1+b_2xQ_2+\right.$$
$$\left.+\left[A\left(\frac{1}{2}\right)A\left(\frac{5}{4}\right)+\frac{1}{2}b_1A\left(\frac{11}{8}\right)\right]xQ_0+b_1A(1)xQ_1+\frac{1}{6}b_1^2xQ_2\right\rangle+$$
$$\left.+c_1\left[3B\left(\frac{4}{3}\right)Q_1+2b_2\left(Q_2-\frac{1}{5}Q_3\right)+b_1^2\left(-\frac{3}{8}Q_1+\frac{1}{3}Q_2-\frac{1}{15}Q_3\right)\right]\right\}$$

$$P=\pi\vartheta(\lambda)[\Theta(\lambda)]^{-1} \tag{2.6}$$

$$\vartheta(\lambda)=c_{-1}+\frac{1}{2}c_1+\frac{3}{8}c_3+\frac{15}{16}c_5-\frac{1}{4\lambda^2}(c_1+c_3)A\left(\frac{3}{2}\right)+\frac{1}{12\lambda^4}c_1b_1A\left(\frac{3}{2}\right)-\frac{1}{\lambda^4}c_1B\left(\frac{4}{3}\right)$$

$$\Theta(\lambda)=\ln 2\lambda+a_0+\frac{1}{\lambda^2}A(1)-\frac{1}{4\lambda^4}\left[A\left(\frac{3}{2}\right)\right]^2+\frac{9}{4\lambda^4}B\left(\frac{7}{6}\right)$$

$$c_{-1}=\lambda+\mu+\frac{\mu^2}{2\lambda}+\frac{\mu^3}{6\lambda^2}+\frac{\mu^4}{24\lambda^3}+\frac{\mu^5}{120\lambda^4}+\frac{\mu^6}{720\lambda^5}$$

$$c_0=\mu+\frac{\mu^2}{\lambda}+\frac{\mu^3}{2\lambda^2}+\frac{\mu^4}{6\lambda^3}+\frac{\mu^5}{24\lambda^4}+\frac{\mu^6}{120\lambda^5}$$

$$c_1=\frac{\mu^2}{2\lambda}+\frac{\mu^3}{2\lambda^2}+\frac{\mu^4}{4\lambda^3}+\frac{\mu^5}{12\lambda^4}+\frac{\mu^6}{48\lambda^5}$$

$$c_2=\frac{\mu^3}{6\lambda^2}+\frac{\mu^4}{6\lambda^3}+\frac{\mu^5}{12\lambda^4}+\frac{\mu^6}{36\lambda^5},\quad c_3=\frac{\mu^4}{24\lambda^3}+\frac{\mu^5}{24\lambda^4}+\frac{\mu^6}{48\lambda^5}$$

$$c_4=\frac{\mu^5}{120\lambda^4}+\frac{\mu^6}{120\lambda^5},\quad c_5=\frac{\mu^6}{720\lambda^5}$$

$$Q_0=-1,\quad Q_1=-x^2+\frac{1}{2},\quad Q_2=-x^4+\frac{1}{2}x^2+\frac{1}{8},\quad Q_3=-x^6+\frac{1}{2}x^4+\frac{1}{8}x^2+\frac{1}{16}$$

$$A(\gamma)=\alpha_1+\gamma b_1-b_1\ln 2\lambda,\quad B(\gamma)=a_2+\gamma b_2-b_2\ln 2\lambda$$

Further, from (1.12) we obtain the forces acting on the punch

$$\frac{Q}{\theta b\delta}=\frac{2\pi}{\lambda}\exp\left(-\frac{2}{\lambda}\right)\left\{P\left[1+\frac{1}{2\lambda}+\frac{3}{16\lambda^2}+\frac{5}{96\lambda^3}+\frac{35}{3072\lambda^4}+\frac{21}{10240\lambda^5}-\right.\right.$$
$$\left.-\frac{1}{32\lambda^4}A\left(\frac{3}{2}\right)\left(1+\frac{1}{2\lambda}\right)\right]+\pi c_0\left\langle\frac{1}{4\lambda}+\frac{1}{8\lambda^2}+\frac{5}{128\lambda^3}+\frac{7}{768\lambda^4}+\frac{7}{4096\lambda^5}+\right.$$
$$+\frac{1}{4\lambda^3}A\left(\frac{5}{4}\right)\left(1+\frac{1}{2\lambda}+\frac{1}{8\lambda^2}\right)+\frac{1}{128\lambda^5}A\left(\frac{4}{3}\right)+\frac{1}{2\lambda^5}\left[\frac{3}{2}B\left(\frac{5}{4}\right)+\frac{3}{8}b_1A\left(\frac{23}{18}\right)+\right.$$
$$\left.\left.+\frac{1}{2}A\left(\frac{1}{2}\right)A\left(\frac{5}{4}\right)\right]\right\rangle+\pi c_1\left(\frac{1}{32\lambda^2}+\frac{1}{64\lambda^3}+\frac{7}{1536\lambda^4}-\frac{b_1}{96\lambda^4}\right)+\pi c_2\left[\frac{3}{16\lambda}+\right.$$
$$\left.\left.+\frac{3}{64\lambda^2}+\frac{1}{32\lambda^3}+\frac{3}{16\lambda^3}A\left(\frac{4}{3}\right)\right]+\pi c_3\frac{1}{32\lambda^2}+\pi c_4\frac{5}{32\lambda}+O\left(\frac{1}{\lambda^6}\right)\right\}\tag{2.7}$$

$$\frac{M_x}{\theta b^3\beta}=\frac{M_y}{\theta b^3\alpha}=\frac{\pi}{\lambda}\exp\left(-\frac{6}{\lambda}\right)\left\{P\left[1+\frac{3}{2\lambda}+\frac{27}{16\lambda^2}+\frac{45}{32\lambda^3}+\frac{945}{1024\lambda^4}+\frac{5103}{10240\lambda^5}-\right.\right.$$
$$\left.-\frac{9}{32\lambda^4}A\left(\frac{3}{2}\right)\left(1+\frac{3}{2\lambda}\right)\right]+\pi c_0\left\langle\frac{3}{4\lambda}+\frac{9}{8\lambda^2}+\frac{135}{128\lambda^3}+\frac{189}{256\lambda^4}+\frac{8505}{20480\lambda^5}+\right.$$
$$+\frac{3}{4\lambda^3}A\left(\frac{5}{4}\right)\left(1+\frac{3}{2\lambda}+\frac{9}{8\lambda^2}\right)+\frac{27}{128\lambda^5}A\left(\frac{4}{3}\right)+\frac{3}{2\lambda^5}\left[\frac{3}{2}B\left(\frac{5}{4}\right)+\frac{3}{8}b_1A\left(\frac{23}{18}\right)+\right.$$
$$\left.\left.+\frac{1}{2}A\left(\frac{1}{2}\right)A\left(\frac{5}{4}\right)\right]\right\rangle+\pi c_1\left(\frac{9}{32\lambda^2}+\frac{27}{64\lambda^3}+\frac{189}{512\lambda^4}-\frac{3b_1}{32\lambda^4}\right)+\pi c_2\left[\frac{9}{16\lambda}+\right.$$
$$\left.\left.+\frac{27}{32\lambda^2}+\frac{27}{32\lambda^3}+\frac{9}{16\lambda^3}A\left(\frac{4}{3}\right)\right]+\pi c_3\frac{9}{32\lambda^2}+\pi c_4\frac{15}{32\lambda}+O\left(\frac{1}{\lambda^6}\right)\right\}\tag{2.8}$$

3. We will now consider the case of small values of the parameter λ, i.e. the case of a relatively wide ring. Here the main term of the asymptotic form of the solution of integral equations (1.11) for small λ must be constructed from solutions of the boundary-layer type, which describe the rapid variability of the contact pressure in the neighbourhood of the contours $r = a$ and $r = b$, and the penetrating (degenerate) solution, which holds far from the contours $r = a$ and $r = b$. This construction can have the multiplicative from [3, 5].

$$q_0(r)=\frac{4\theta\delta}{\pi^2\sqrt{b^2-r^2}}\left(\frac{a}{\sqrt{r^2-a^2}}+\arccos\frac{a}{r}\right),\quad \frac{Q}{\theta b\delta}=4\tag{3.1}$$

$$\frac{q_1(r)}{\alpha}=\frac{\tilde{q}_1(r)}{\beta}=\frac{8\theta r}{\pi^2\sqrt{b^2-r^2}}\left[\frac{a}{\sqrt{r^2-a^2}}\left(1-\frac{a^2}{3r^2}\right)+\arccos\frac{a}{r}\right],\quad \frac{M_y}{\theta b^3\alpha}=\frac{M_x}{\theta b^3\beta}=\frac{8}{3}\tag{3.2}$$

or the additive from [3, 5]

$$q_0(r)=\frac{2\theta\delta}{\pi\sqrt{b^2-r^2}}\left(1-\frac{1}{\pi}\arccos\frac{1+\varepsilon^2-2\eta^2}{1-\varepsilon^2}\right)+\frac{2\theta\delta}{\pi^2\sqrt{r^2-a^2}}\ln\frac{1+\varepsilon}{1-\varepsilon}$$

$$\frac{Q}{\theta b\delta}=4\left(\frac{2}{\pi}\arccos\varepsilon+\frac{1}{\pi}\sqrt{1-\varepsilon^2}\ln\frac{1+\varepsilon}{1-\varepsilon}\right) \tag{3.3}$$

$$\frac{q_1(r)}{\alpha}=\frac{\tilde{q}_1(r)}{\beta}=\frac{4\theta r}{\pi\sqrt{b^2-r^2}}\left(1-\frac{1}{\pi}\arccos\frac{1+\varepsilon^2-2\eta^2}{1-\varepsilon^2}\right)+$$
$$+\frac{4\theta r}{\pi^2\sqrt{r^2-a^2}}\left[\ln\frac{1+\varepsilon}{1-\varepsilon}+\frac{b^2}{r^2}\left(\frac{1-\varepsilon^2}{2}\ln\frac{1+\varepsilon}{1-\varepsilon}-\varepsilon\right)\right]$$

$$\frac{M_y}{\theta b^3\alpha}=\frac{M_x}{\theta b^3\beta}=\frac{8}{3}\left[\frac{2}{\pi}\arccos\varepsilon-\frac{1}{2\pi}\varepsilon\sqrt{1-\varepsilon^2}+\frac{1}{4\pi}(5+\varepsilon^2)\sqrt{1-\varepsilon^2}\ln\frac{1+\varepsilon}{1-\varepsilon}\right] \tag{3.4}$$

In (3.3) and (3.4) we have introduced the notation $\varepsilon = a/b$ and $\eta = a/r$, and the expressions for Q, M_y and M_x were obtained from (1.12).

If the inclination of the punch is due to the fact that the applied force Q is off-centre, as shown in Fig. 1, and it has no initial inclination, then, from (2.6) and (2.7) or (3.1) and (3.2) or (3.3) and (3.4) we can formulate the condition for the foot of the punch not to lose contact with the base. For example, using (3.1) and (3.2) with $\beta = 0$, $M_x = 0$ and $\tilde{q}_1(r) \equiv 0$ we obtain

$$e\leqslant b\left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}+\arccos\varepsilon\right)\left[\frac{\varepsilon(3-\varepsilon^2)}{\sqrt{1-\varepsilon^2}}+3\arccos\varepsilon\right]^{-1} \tag{3.5}$$

where e is the eccentricity of the application of the force Q (its distance from the axis of symmetry of the punch).

4. We will describe in more detail one more method of constructing the asymptotic solution of the problem for small λ, which was mentioned in [4].

We will start from the integral equations (2.3). Note that the kernels $M_m(t)$ ($m = 0, 1$) of these equations can be written, by virtue of (1.8), in the form

$$M_m(t)=\int_0^\infty\frac{L_m(s)}{s}\cos st\,ds \tag{4.1}$$

If we further represent the meromorphic function $L_m(s)$ in the form of the principal values [10], we can obtain the following expansion for the kernels

$$M_m(t)=\pi\sum_{n=0}^{\infty}\frac{(2n-1+2m)!!(2n-1)!!}{(2n+2m)!!(2n)!!}\exp\left[-\left(2n+m+\frac{1}{2}\right)|t|\right] \tag{4.2}$$

We will first consider the first equation of (2.3). It can be shown that the sum of the functions $\zeta_1[(1+x)/\lambda]$ and $\zeta_2[1-x)/\lambda]$, which yields a solution of the system of integral equations

$$\int_{-1}^{\infty}\zeta_1\left(\frac{1+\xi}{\lambda}\right)M_0\left(\frac{x-\xi}{\lambda}\right)d\xi=\int_{-\infty}^{-1}\zeta_2\left(\frac{1-\xi}{\lambda}\right)M_0\left(\frac{x-\xi}{\lambda}\right)d\xi\quad(-1\leqslant x<\infty) \tag{4.3}$$
$$\int_{-\infty}^{1}\zeta_2\left(\frac{1-\xi}{\lambda}\right)M_0\left(\frac{x-\xi}{\lambda}\right)d\xi=\pi\lambda\exp\left(\frac{1+x}{2\lambda}\right)+\int_1^{\infty}\zeta_1\left(\frac{1+\xi}{\lambda}\right)M_0\left(\frac{x-\xi}{\lambda}\right)d\xi\quad(-\infty<x\leqslant 1)$$

will also be a solution of the first equation of (2.3), i.e.

$$p_0(x) = \zeta_1\left(\frac{1+x}{\lambda}\right) + \zeta_2\left(\frac{1-x}{\lambda}\right) \tag{4.4}$$

Hence, the problem reduces to finding the asymptotic solution for small λ of the system of equations (4.2).

By obvious changes of the variables we can convert Eqs (4.3) to the form

$$\begin{aligned}&\int_0^\infty \zeta_1(\tau)M_0(t-\tau)d\tau = \int_{2/\lambda}^\infty \zeta_2(\tau)M_0\left(t+\tau-\frac{2}{\lambda}\right)d\tau \quad (0 \leqslant t < \infty)\\&\int_0^\infty \zeta_2(\tau)M_0(t-\tau)d\tau = \pi e^{1/\lambda}e^{-t/2} + \int_{2/\lambda}^\infty \zeta_1(\tau)M_0\left(t+\tau-\frac{2}{\lambda}\right)d\tau, \quad (0 \leqslant t < \infty)\end{aligned} \tag{4.5}$$

From the meaning of the problem $\zeta_1(t) \sim e^{-\kappa t}$, $\kappa > 0$ as $t \to \infty$. Hence, the integral on the right-hand side of the second equation of (4.5) for small λ is exponentially small and, in the zeroth (main) approximation, it can be neglected compared with the first term on the right-hand side. Then, to determine $\zeta_2(t)$ we have a Wiener–Hopf integral equation [11], where the function $L_0(s)$ of the form (1.8) is easily factorized. As a result we have [4]

$$\zeta_2(t) = 2\pi^{-1}e^{1/\lambda}e^{-3t/2}(1-e^{-2t})^{-1/2} \tag{4.6}$$

Substituting (4.6) into the first equation of (4.5) and evaluating the integral on the right-hand side using expansion (4.2) we again arrive at a Wiener–Hopf equation in the function $\zeta_1(t)$

$$\int_0^\infty \zeta_1(\tau)M_0(t-\tau)d\tau = e^{-2/\lambda}\sum_{n=0}^\infty \left[\frac{(2n-1)!!}{(2n)!!}\right]^2 \frac{1}{n+1}e^{-t(2n+1/2)} \quad (0 \leqslant t < \infty) \tag{4.7}$$

Its solution has the form [4]

$$\zeta_1(t) = \frac{2}{\pi^2}e^{-2/\lambda}e^{-3t/2}(1-e^{-2t})^{-1/2}\sum_{n=0}^\infty \frac{(2n-1)!!}{(n+1)(2n)!!}F\left(-n,1,\frac{1}{2},1-e^{-2t}\right) \tag{4.8}$$

Hence, the principal term of the asymptotic form of the solution of the first integral equation of (2.3) for small λ is given by (4.4), (4.6) and (4.8).

From the first formula of (1.12) we obtain

$$\begin{aligned}\frac{Q}{\theta b\delta} &= \frac{2\pi}{\lambda}e^{-2/\lambda}\int_{-1}^1 p_0(\xi)e^{(1+\xi)/2\lambda}d\xi = 4\left[\sqrt{1-\varepsilon^2} + \frac{1}{\pi}\varepsilon^2 \arccos\varepsilon + \right.\\&\left. + \frac{1}{\pi}\varepsilon^3\sqrt{1-\varepsilon^2}\sum_{n=1}^\infty \frac{(2n-1)!!}{(n+1)(2n)!!}F\left(-n+1,1,\frac{3}{2},1-\varepsilon^2\right)\right]\end{aligned} \tag{4.9}$$

Here we have used the relation [12]

$$\int(1-x)^{-1/2}x^{-1/2}F\left(-n,1,\frac{1}{2},x\right) = 2x^{1/2}(1-x)^{1/2}F\left(-n+1,1,\frac{3}{2},x\right) \tag{4.10}$$

In exactly the same way we can construct, for small λ, the principal term of the asymptotic form of the second integral equation of (2.3)

$$p_1(x) = \zeta_1\left(\frac{1+x}{\lambda}\right) + \zeta_2\left(\frac{1-x}{\lambda}\right), \quad \zeta_2(t) = 4\pi^{-1}e^{3/\lambda}e^{-5t/2}(1-e^{-2t})^{-1/2}$$

$$\zeta_1(t) = \frac{4}{\pi^2} e^{-2/\lambda} e^{-5t/2} (1 - e^{-2t})^{-1/2} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(n+2)(2n)!!} F\left(-n, 1, \frac{1}{2}, 1 - e^{-2t}\right)$$

$$\frac{M_y}{\theta b^3 \alpha} = \frac{M_x}{\theta b^3 \beta} = \frac{8}{3}\left[\sqrt{1-\varepsilon^2} + \frac{1}{2}\varepsilon^2\sqrt{1-\varepsilon^2} + \frac{3}{4\pi}\varepsilon^4 \arccos\varepsilon + \right. \tag{4.11}$$

$$\left. + \frac{3}{2\pi}\varepsilon^5\sqrt{1-\varepsilon^2} \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(n+2)(2n)!!} F\left(-n+1, 1, \frac{3}{2}, 1-\varepsilon^2\right)\right]$$

In conclusion we will give some numerical results from the quantities $g_1 = Q/(\theta b \delta)$ and $g_2 = M_y/(\theta b^2 \alpha) = M_x/(\theta b^3 \beta)$, obtained from the formulae derived in this paper

λ	1/2	1	2	4	8
$g_1(2.7)$		3.99	3.97	3.86	3.63
$g_1(3.1)$	4.00	4.00	4.00		
$g_1(3.3)$	4.00	4.00	3.96		
$g_1(4.9)$	4.00	4.00	3.96		
$g_2(2.8)$			2.64	2.64	2.53
$g_2(3.2)$	2.67	2.67	2.67	2.67	
$g_2(3.4)$	2.67	2.67	2.66	2.63	
$g_2(4.11)$	2.67	2.67	2.66	2.64	

It can be seen that the asymptotic solutions for large and small λ of the first equation of (1.11) join up when $\lambda \in [1, 2]$, while the second and third equations of (1.11) join up when $\lambda \in [2, 4]$.

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